Spontaneous Symmetry Breaking

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Spontaneous Symmetry Breaking is one of the fundamental phenomena in field theory and condensed matter systems. Initially, a system in a ground state respects symmetry. Some changes in the system could lead to new accessible ground states which do not respect the original symmetry present in the system. We say the symmetry is broken. This loss of symmetry has consequences - a particle's spectrum in the newly found vacuum could be entirely different than the spectrum in the original unbroken state.

I. SYMMETRY BREAKING: A CLASSICAL EXAMPLE

A classical example of symmetry breaking would be a stick that buckles under load. The process directly connects to the field theory. A stick is erect and unbuckled under zero load. What is the symmetry of the system? - It is rotationally invariant around the axis of the stick. If now the load is slowly increased the stick will eventually buckle at a critical value. The symmetry present in the original state is now broken.

I would like to add a few things here. Initially, there are fluctuations in the force applied. It's just that those fluctuations are too small to cause any change in the system. The system is invariant in its lowest energy configuration. This will be true as long as those fluctuations are small (because the load is small). Now as the load grows fluctuations also grow and that results in the buckling. The point to note here is that after a critical load value an infinite number of lower energy states appear and the system goes into one of them. The rotational symmetry of the system is lost and all those lower energy states are related to each other by rotation. We will see parallels of this simple example in all the systems we study in this report.

II. SYMMETRY BREAKING IN FERROMAGNETS

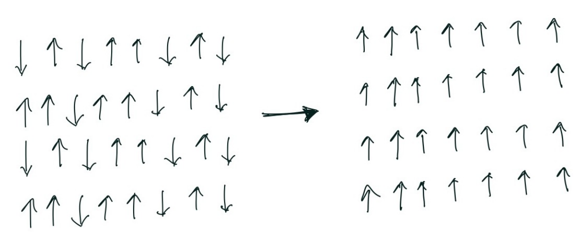

Figure of the left depicts depicts magnetic moments when $T > T_c$, and figure on right depicts spins when $T < T_c$

A ferromagnet has a large number of magnetic moments. These moments are randomly aligned at room temperature as shown in the figure. [1] If we calculate the Magnetization (Magnetic dipoles per unit volume, $\vec{M}$) of the system[2],

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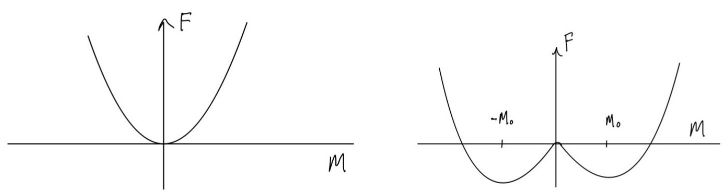
[1] Actually the situation is a little more complex: moments tend to align themselves to the neighboring moments, so a naturally low-energy state is when all the moments are in the same direction. But this happens in small patches in the magnet. A patch's magnetization could be in a certain direction and another's could be in some other's direction. Our analysis still works because there are a large number of these patches aligned in all sorts of directions

[2] We take the spherical average of magnetic dipole of a volume in the material such that the Radius of the sphere is very large compared to the dimension of the patches

we will get zero. since the average number of magnetic moments aligned in a particular direction is equal to the average number of magnetic moments aligned in a completely opposite direction. There is a symmetry present in the system - Magnetization is unchanged with the 180 flip of all the moments. It is experimentally known that as the temperature is lowered, and after a critical value T_c all the moments are aligned in one of the two configurations possible; magnetization $\pm M_0$, and the original symmetry ($S \to -S$, for all i) of the system is lost.

$$F = A_0 + A_2 M^2 + A_4 M^4 + ... \tag{1}$$

We could write the free energy ($F = U - TS$) of the system in the powers of M, with the assumption that the first few terms in the series give a good approximation of the true value. If $M \to -M$ then $F \to F$. This is true in the range $T > T_c$. To understand how the system changes when the temperature is lowered let A_2 in the expansion be such that it changes sign as $T < T_c$ or A_2 is proportional to $(T - T_c)$. Now the expression becomes $F = A_0 - A_2 M^2 + A_4 M^4 ...$ for $(T < T_c)$. If we plot F in the two limits; $T > T_c$, and $T < T_c$ we see that in the case $T < T_c$ the system has two available vacuums, and the system can now move into either one of them with the new minimum at $\pm M_0$.The original symmetry present is robbed from the system.



Free energy before and after Symmetry Breaking

CAVEATS: 1. If we considered this system quantum mechanically there is a probability for transitioning of the system from one minimum to another minimum. 2. When there is an infinite number of degrees of freedom like in a field theory tunneling then tunneling probability is suppressed and SSB can happen. 3. This is just an example to show how symmetry is lost with newly available vacuums. We are not expecting any massless excitations, as we will later see that for massless excitations there should be continuous symmetry present.

III. CONTINUOUS SPONTANEOUS SYMMETRY BREAKING AND GOLDSTONE THEOREM

Now we discuss a slightly non-trivial case of continuous symmetry breaking in field theory. I begin with writing a Lagrangian, which is the sum of the two scalar fields Lagrangian with the interaction as shown. Note that the mass term in this Lagrangian is flipped to account for the breaking of symmetry.

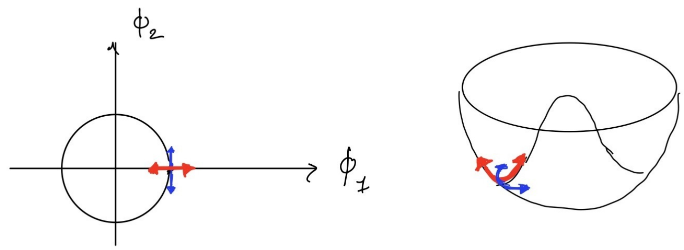
$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2] + \frac{\mu^2}{2}[\phi_1^2 + \phi_2^2] - \frac{\lambda}{4}[\phi_1^2 + \phi_2^2] \tag{2}$$

This Lagrangian is symmetric under the transformation shown below. We say that Lagrangian is symmetric under rotations in $\phi_1 - \phi_2$ plane with infinite minima or vacuams at every value of ϕ_1 and ϕ_2 that follows the equation $\phi_1^2 + \phi_2^2 = \frac{6\mu^2}{\lambda}$.

$$\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} \to \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} \begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} \tag{3}$$

$$V(\phi_1, \phi_2) = -\tfrac{\mu}{2}[\phi_1^2 + \phi_2^2] + \tfrac{\lambda}{4}[\phi_1^2 + \phi_2^2]$$

The potential when drawn looks like a Mexican hat. An important point to note here is that symmetry is still present in this equation, it is what I plan to do next that will break the symmetry - we will look for excitations in the fields [3] The way we do is by first choosing a vacuum among infinite vacuums. [4] We chose $\phi_1 = \sqrt{\frac{6\mu^2}{\lambda}}$, and $\phi_2 = 0$. Now we expand the potential V around this chosen vacuum with $\phi \to \phi - \sqrt{\frac{6\mu^2}{\lambda}}$. Let $\phi_0 = \sqrt{\frac{6\mu^2}{\lambda}}$



We are expanding about a vacuum here

$$V(\phi_1 - \phi_0, \phi_2 - 0) = V(\phi_0, 0) + \frac{\partial V(\phi_0, 0)}{\partial \phi_1}(\phi_1 - \phi_0) + \frac{\partial V(\phi_0, 0)}{\partial \phi_2}(\phi_2 - 0) \quad (4)$$

$$+\frac{1}{2}[\frac{\partial^2 V}{\partial \phi_1^2}(\phi_1 - \phi_0)^2 + \frac{\partial^2 V}{\partial \phi_2^2}(\phi_2 - \phi_0)^2 + 2\frac{\partial^2 V(\phi_0, 0)}{\partial \phi_1 \phi_2}(\phi_1 - \phi_0)(\phi_2 - 0)] + ... \quad (5)$$

where on substitution of the derivatives and defining $\phi_1\prime = \phi_1 - \phi_0$, and $\phi_2\prime = \phi_2$ we get:

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi_1')^2 + (\partial_\mu \phi_2')^2] + \mu^2 \phi_1^{2\prime} + \mathcal{O}(3) \quad (6)$$

We started with equation 2 and ended up with equation 6. What to make of this change? We know that term in the square in fields is called a mass term because square root of the coefficient of that term gives the mass of the excitation i.e the mass of the particle associated with that field.

Question: What happened to the mass of the particle and how to interpret it?
Answer: On comparison of the two Lagrangians we see that the mass of the excitations of ϕ_1 field (particles) about the new vacuum is $\sqrt{2}\mu$ instead of the original μ, and there is no excitation of the ϕ_2 field about this new vacuum. We observe that because of symmetry breaking mass of the ϕ_1 field's particle is now $\sqrt{2}\mu$ and ϕ_2 particle's mass is zero.

One way to think about what happened here is to look at the potential of this Lagrangian. There is a gutter that is formed from the locus of points of the minima. The expansion that resulted in equation 6 meant small movements along ϕ_1 and ϕ_2 directions, but from the figure of the potential, the expansion along ϕ_1 amount to movement along the radial direction and hence that movement experiences "resistance" and therefore we have the mass for ϕ_1 field, whereas any movement along ϕ_2 results the movement along the gutter remaining at the same potential, hence no "resistance", so no mass for ϕ_2 after symmetry breaking.

[3] Since excitations in the fields are the particles and we want to look at what dispersion relation looks like for vacuum excitation. And we will compare the dispersion relations of the excitation before and after the symmetry breaking
[4] We can always do that since all those vacuums are identical

We can condense what we accomplished here into a theorem: **GOLDSTONE THEOREM**: *If a continuous symmetry is broken, it will result in massless excitation called **Goldstone Mode**, and the massless particle is called **Goldstone Boson***

IV. SSB IN COMPLEX SCALAR FIELDS WITH GLOBAL AND LOCAL SYMMETRIES (HIGGS MECHANISM)

The plan for this section is to see two examples of symmetry breaking: first, we will see how symmetry breaking plays out in a theory with global symmetry, and later in a local symmetry or Higgs mechanism.

Consider a Lagrangian with a positive mass term and global symmetry.

$$\mathcal{L} = (\partial_\mu \Psi)^\dagger (\partial_\mu \Psi) + \mu^2 \Psi^\dagger \Psi - \lambda(\Psi^\dagger \Psi)^2 \tag{7}$$

This is symmetric under a global transformation $\Psi \to \Psi \exp(i\alpha)$ which in polar coordinates looks like

$$\Psi(x) = \rho(x) \exp(i\theta(x))$$

which translates to symmetry under $\rho \to \rho$, and $\theta \to \theta + \alpha$. The Lagrangian in polar coordinates is

$$\mathcal{L} = (\partial_\mu \rho)^2 + \rho^2 (\partial \mu \theta)^2 - \lambda(\rho)^4 + \mu^2 (\rho)^2 \tag{8}$$

Going through the same procedure we get the lagrangian (ignoring interaction) which is expanded about the point $\rho_0 = \sqrt{\mu^2/2\lambda}$, and $\theta_0 = 0$

$$\mathcal{L} = (\partial_\mu \rho')^2 - 2\mu^2 \rho'^2 + \mu^2 \rho^2 - 4(\frac{\mu^2 \lambda}{2})^{\frac{1}{2}} \rho'^3 + \frac{\mu^2 \lambda}{2} (\partial \mu \theta')^2 + ... \tag{9}$$

On a direct comparison of equations 8 and 9, we see that the expansion along the gutter resulted in mass excitations of θ fields, and expansion along the radius results in excitation in ρ fields, but with mass $\sqrt{2}\mu$

Now let us demand that our Lagrangian respects local symmetry $\Psi \to \Psi\alpha(x)$. But if you make that transformation you'd find that the Lagrangian is not symmetric. The consequence of the demand is that we would have to introduce a gauge field to compensate for the local variation of the field. We do this by including gauge fields A_μ using a covariant derivative which is defined as $\mathcal{D}_\mu = \partial_\mu + iqA_\mu(x)$ where A_μ transforms as $A_\mu \to A_\mu - \frac{i}{q}\partial_\mu$

Our Lagrangian is

$$\mathcal{L} = (\partial_\mu \Psi^\dagger - iqA^\mu \Psi^\dagger)(\partial_\mu \Psi + iqA_\mu \Psi) + \mu^2 \Psi^\dagger \Psi - \lambda(\Psi^\dagger \Psi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{10}$$

Let us note important features of this Lagrangian. In addition to what is described above, we can see that the Ψ field is massive with mass μ, and A_μ field is massless. Now as in the case of global symmetry we use the polar coordinates $\Psi(x) = \rho(x) \exp(i\theta(x))$. Now we can do a variable change such that $\mathcal{C} = A_\mu + \frac{1}{q}\partial_\mu \theta$, and this is also gauge invariant. In terms of this substitution, Lagrangian becomes:

$$\mathcal{L} = (\partial_\mu \rho)^2 + \rho^2 q^2 C^2 - \lambda \rho^4 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \mu^2 \rho^2 \tag{11}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$
Now we do the usual procedure of breaking the symmetry by expanding around a chosen vacuum. We choose $\rho_0 = \sqrt{\frac{\mu^2}{2\lambda}}$, and $\theta_0 = 0$. With change of variable $\rho = \rho_0 + \xi$ we have with $M = q\sqrt{\frac{\mu^2}{\lambda}}$

$$\mathcal{L} = \frac{1}{2}(\partial \mu \xi)^2 - \mu^2 \xi^2 - \sqrt{\lambda} \mu \xi^3 - \frac{\lambda}{4} \xi^4 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{M^2}{2} C^2 + q^2 \frac{\mu^2}{\lambda}^{\frac{1}{2}} \xi C^2 + \frac{1}{2} q^2 \xi^2 C^2 + ... \tag{12}$$

As we have done previously we compare the Lagrangians before and after symmetry breaking. On comparing equations 11 and 12 we see:
1. The mass of ξ field excitation is still $\sqrt{2}\mu$. However, initially we had Ψ and $\Psi^\dagger$ as massive scalar excitations.
2. Now we have mass excitations in the C_μ field. C_μ is a massive gauge field. It has three polarization hence the excitations are spin-1 massive particles. Originally A_μ was massless and only had two polarizations.
3. θ field is completely missing.

we can also note that the degrees of freedom are conserved, since before and after the SSB we have the same number of modes.